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List of recommened Exercises II

Module 2

501. Show that the process $\tilde{B}_t = B(t+T) - B(T)$ is a standard Brownian motion. (Hint: Check the covariance condition)

502. Let $X_t = \mu dt + \sigma dB_t$ be a be a Brownian motion with drift, where $\mu, \sigma \neq 0$, Let $X_0 = x \in (a,b)$. Use Dynkin's formula to show that the expected exit time of X_t from the interval (a,b) is finite.

503. Let X_t be a geometric Brownian motion with

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \quad X_0 = x \in (a,b)$$

Convince yourself that the expected exit time of X_t from the interval (a,b) is finite. (Distinguish the case where $\mu - \frac{1}{2}\sigma^2 = 0$ and $\mu - \frac{1}{2}\sigma^2 \neq 0$).

504. Characterise all harmonic functions $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ of the form $u(x) = h(||x||)$.

505. Suppose that $r > 0, a_j > 0, b_j, c_j \in \mathbb{R}$ for $j = 1, \dots, n$. Assume that $\sum_{j=1}^n c_j < r$. Consider the set

$$D = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{j=1}^n (a_j x_j^2 + b_j x_j + c_j) < r\}.$$

Let B_t be an n-dimensional Brownian motion starting at $(0, \dots, 0) \in \mathbb{R}^n$. Determine the expected exit time of B_t from D .

506. Let $B_t \in \mathbb{R}^2$ with $B_0 = x$ where $r < ||x|| < R$, i.e. x is in the two-dimensional annulus $A(r,R)$. Determine the probability that B_t hits r before R (or, the probability that B_t exits the annulus from r).

Module 3

The exercises in §6 are quite elementary, nonetheless, they are helpful in terms of understanding the material. You can skip all of them if you are familiar with the heat equation.

601. **(warm up)** Show that $f * g = g * f$.

The following three exercises consider properties of the heat kernel:

602. Let $g(t,x)$ be the fundamental solution to the 1d heat equation, show that $g_t = \Delta g$.

603. Let $g(t,x)$ be the fundamental solution to the 1d heat equation, convince yourself that

$$\lim_{t\searrow 0} g(t,x) = \begin{cases} 0, x \neq 0, \\ \infty, x = 0. \end{cases}$$

604. Let $g(t,x)$ be the fundamental solution to the 1d heat equation, fix $t > 0$, show that

$$\int_{\mathbb{R}} g(t,x)dx = 1.$$

(Hint: you can use the property of the Gaussian distribution)

(Same problem in higher dimension) Let $g(t,x)$ be the fundamental solution to the 1d heat equation in $\mathbb{R}^n$ (F6. Thm. 6.7), fix $t > 0$, show that

$$\int_{\mathbb{R}^n} g(t,x)dx = 1.$$

The following exercises consider properties of the solutions to the heat equation, we will sum them up in F7.

(Setting for 6.5-6.9) We consider the one-dimensional heat equation

$$u_t(t,x) \stackrel{*}{=} u_{xx}(t,x),$$

where

$$0 < t < T, T \in (0,\infty]. a < x < b, a \in [-\infty,\infty), b \in (-\infty,\infty].$$

605. **(Linearity)** Show that if u, v both solve $(*)$, and $\alpha, \beta \in \mathbb{R}$, then $\alpha u + \beta v$ is also a solution.

606. **(Shift and Scale)** Show that if u solves $(*)$, and $\alpha > 0, x_0 \in \mathbb{R}$, then $u(\alpha^2 t, \alpha x - x_0)$ solves $(*)$ for $t \in (0, \alpha^2 T)$ and $x \in (\alpha a + x_0, \alpha b + x_0)$.

607. **(Differential property)** Show that if $u \in C^3$ and u solves $(*)$, then u_t, u_x also solve $(*)$.

608. **(Integration)** If u solves $(*)$ and $v(t,x) = \int_a^x u(t,z)dz, x \in (a,b)$, then v also solves $(*)$ given that

$$\lim_{z\to a} u_x(t,z) = 0$$

for each $t \in (0,T)$.

609. **(Convolution)** If u solves $(*)$ and $f : \mathbb{R} \to \mathbb{R}$ is a function, then $(u * f)(t,x)$ also solves $(*)$.

701. **(warm up)** Show that if f is even and h is odd relative to x_0, then $f * h$ is odd relative to x_0.

702. Let $u(t,x)$ be a solution to the heat equation $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$ on $\{(t,x) : t > 0, x > 0\}$ with $u(0,x) = u_0(x)$ for $x > 0$, and $\frac{\partial u}{\partial x}(t,0) = 0$ for $t > 0$. Show that

$$u(t,x) = \int_0^\infty u_0(y)h(t,x,y)dy$$

for some function $h(t,x,y)$.

703. **(Time varying boundary)** Check that

$$u(t,x) = \int_0^t G_y(t-s,x,0)f(s)ds$$

solves the quarter-plane problem

$$\begin{cases} u_t - u_{xx} = 0, & t,x > 0, \\ w(t,0) = f(t), \\ w(0,x) = 0, \end{cases}$$

where $G(t,x,y)$ is the Green's function for the quarter-plane problem.

704. **(Infinite propagation speed)** Show that in the Cauchy IVP problem, if the initial data $\Phi(x) \geq 0, \Phi(x) \not\equiv 0$, then

$$u(t,x) = \int_{\mathbb{R}} g(t,x-y)\Phi(y)dy > 0$$

for all $x \in \mathbb{R}, t > 0$. (*Hint: Strong maximum principle, use minimum instead*)

705. **(HE with positive rate of diffusion)** Show that

$$g\,(t,x) \;=\; \frac{1}{\sqrt{2kt}}\varphi(\frac{x}{\sqrt{2kt}})$$

solves $g_t = kg_{xx}$.

801. Verify that $\rho(t,x) = \frac{1}{\sqrt{2\pi t}}\exp(-\frac{x^2}{2t})$, the probability density function of a standard BM at time t, solves $\rho_t = \frac{1}{2}\rho_{xx}$.

802. Let $u(t,x)$ be the probability density function of a stochastic process at time t and solves $u_t = u_{xx}$. What is this process?

803. **(warm up)** Write down $\mathcal{L}^*$ for the following 1d Ito diffusions:

(a) $dX_t = dB_t$,

(b) $dX_t = \mu dt + \sigma dB_t$,

(c) $dX_t = \mu X_t dt + \sigma dB_t$,

(d) $dX_t = \mu X_t dt + \sigma X_t dB_t$.

804. Show that $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$ with respect to the quadratic inner product: let g, h be smooth and vanish at infinity (along with their derivatives), then

$$\int_{\mathbb{R}} h(x)(\mathcal{L}g)(x)dx = \int_{\mathbb{R}} g(x)(\mathcal{L}^*h)(x)dx.$$

805. Convince your self that the steady-state distribution we find in F8 is indeed the limiting distribution of the OU process. (Hint: lifting T to infinity)

806. **(BM has no steady-state distribution)** Write down the KFE for a standard BM, and

    (a) show that the stationary solution is of the form $\rho_\infty(x) = ax + b$,

    (b) look at the boundary condition at ∞ and conclude $\rho_\infty(x) = b$,

    (c) conclude that such a distribution cannot be normalised.

    (This implies that (as in Kohn section 1), $\rho_\infty(x) = 0$. This agrees with our knowledge of the Gaussian hump with always-growing width and shrinking height)

807. **(BM in the unit interval)** Now we look at something similar to 10.4. Consider the KFE of a 1d Brownian motion in the interval $(0, 1)$ with the boundary condition

$$\frac{\partial \rho_\infty}{\partial x}|_{x=0} = \frac{\partial \rho_\infty}{\partial x}|_{x=1} = 0.$$

    Show that the steady-state distribution exists and is uniform.
    (Remark. This homogeneous Neumann boundary condition here is what we call the 'reflecting boundary'. Namely the process is reflected when it hits the boundary. We will not go too deeply in that and you probably have seen it somewhere else. Other types of boundary conditions for KFE are 'absorbing bdr', 'periodic bdr', 'sticky bdr'...)

808. **(Moments of BM again)** Fix $t > 0$ and let $\beta_k(t) := \mathbb{E}[B_t^k]$. Recall that

    (a) $\mathbb{E}[X^n] = \int_{\mathbb{R}} x^n f(x)dx$ where f is the probability density function,

    (b) $\rho(t, x)$ decays sufficiently fast at infinity.

    Use the KFE of the standard BM to show that

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds.$$

809. **(Generalised OU)** Let $a > 0$, $V : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} e^{-V(x)}dx < \infty.$$

    Find the limiting density function for X_t that solves

$$dX_t = -aV'(X_t)dt + \sqrt{2a}dB_t.$$

    (Remark. why is it not needed that we specify the initial value of Connect it with what you know about Markov processes.)